Dimensions in Mathematics

Table of Contents

VOLUME 35 NUMBER 2 FALL 2015

Dimensions in Mathematics is published by the Florida Council of Teachers of Mathematics (FCTM) twice yearly. Subscribers have permission to reproduce any classroom activities published in Dimensions.

Dimensions in Mathematics is devoted to the improvement of mathematics education at all instructional levels. Articles which appear in ***Dimensions in Mathematics*** present a variety of viewpoints which, unless otherwise noted, should not be interpreted as official positions of the Florida Council of Teachers of Mathematics.

For more information on FCTM, including membership information, visit the FCTM web site www.fctm.net.

Dimensions in Mathematics
Journal of the Florida Council of Teachers of Mathematics

SUBMISSION OF MANUSCRIPTS

Manuscripts for the journal should be double-spaced for 8½" x 11" paper. Send manuscripts as attachments from Microsoft Word via E-mail Illustrations should be camera ready and saved in .gif or .jpg format. Please include a short biographic statement about you and any co-authors. Include your E-mail address and a telephone number. Articles should be no longer than 10 pages double-spaced. Send submissions to:

Rick Austin,
austin@usf.edu

MEMBERSHIP

Send Change of Address to:

Diane Gard
PO Box 411884
Melbourne, FL 32941-1884
gard.diane@brevardschools.org

The **Florida Council of Teachers of Mathematics** publishes ***Dimensions in Mathematics***. Groups affiliated with the National Council of Teachers of Mathematics are granted permission to reprint any article appearing in *Dimensions* provided that one copy of the publication in which the material is reprinted is sent to the Editors and that *Dimensions* is clearly cited as the original source.

The opinions presented in the articles published in *Dimensions* are those of the authors and do not necessarily reflect the position of the Florida Council of Teachers of Mathematics.

From the President's Desk . . .

In this edition of our journal you will definitely find articles that will be useful for you in your various roles in education. I admonish you to take the time to read it, and use the resources as needed. I also encourage you to visit our www.fctm.net website for the most current and up to date information, from election results to grant and award winners. Congratulations to Joyce Hawkins for being selected the 2015 Kenneth Kidd Mathematics Educator of the Year. This honor is well deserved. Joyce's hard work and dedication to mathematics education has not gone unnoticed.

To all of you it has been an honor and a privilege to serve as your president. These past few years have been definitely a growing experience for me. Together I feel we all have grown, I as an individual, and FCTM as an organization. From receiving a national award for this *Dimensions in Mathematics* journal, to consistently presenting stellar conferences, to recently the Board of Directors of FCTM providing sessions focusing on the Florida Standards. Kudos to the board members who stepped up and presented sessions at this past 2015 state conference at the grade brands of K-2, 3-5, 6-8, Algebra I & 2, and Geometry. The sessions were high quality and beneficial to all who attended. These are only a few examples of what we together have been able to accomplish. This journey has definitely been favorable. Your support has been felt and appreciated. I encourage you to join me in looking forward to what is next for FCTM as the journey continues…

Margaret J. Walker

FCTM President 2014 - 2015

Congratulations Joyce Hawkins

Joyce Hawkins (left) with Margaret Walker, President, at the 2015 FCTM Annual Conference.

2015 Kenneth P. Kidd Mathematics Educator of the Year

Joyce Hawkins has been in education for more than 40 years. Joyce received a Bachelor's of Science degree, with major work in Psychology and Mathematics Education from the Florida A&M University. She later earned a Master's of Education degree in Counseling Education from Florida Atlantic University. For most of her professional life Joyce served as a mathematics teacher and department chair. Although officially retired, she is currently serving as a mathematics teacher at the Conservatory Prep High School, Davie FL.

The list of other teaching related activities in which Joyce has participated is extensive. These include; facilitating a network for Advanced Placement Calculus Teachers and for Vertical Teaming, facilitated workshops on how to use the TI-84 calculator, facilitated workshops for teachers to increase their understanding of State Standards for mathematics education and high stakes testing. She has worked with at risk teachers in preparation to pass the state mandated tests for certification. In addition, she has served as a Supervising Teacher for mathematics student teachers. Joyce has presented at our annual conference on the measurement strand of the Sunshine State Standards, and has served on various state committees for the FCAT and for the mathematics teacher certification exam.

Joyce Hawkins has been an active FCTM board member for many years, and has been reappointed as the elections chair annually. If you have voted at the annual conference, then you have encountered Joyce as you cast your ballot. She continues to be a member of the Broward Council of Teachers of Mathematics as well as several other associations. She is a member of: National Council of Teachers of Mathematics, Delta Kappa Gamma, Florida Council of Teachers of Mathematics, the Florida Mathematics Presidential Awardees Association, to name a few. There are others…

The Kenneth Kidd Mathematics Educator of the Year award is the highest honor that FCTM bestows on one of its members. Individuals are nominated for the award by their peers. In order to be eligible, the nominee must have been actively involved in mathematics education in Florida for at least ten years. Details, a nomination form, and a list of past recipients can be found on page 34 of this issue.

Schakow, Joy B. and Stephanie Cugini
Algebra Nation: A Free Resource for Florida Teachers and Students
Dimensions in Mathematics
Fall 2015, 35 (2), 6 – 12.

Algebra Nation: A Free Resource for Florida Teachers and Students

Joy B. Schackow and Stephanie Cugini

Florida Algebra 1 teachers know that the Algebra 1 Florida Standards Assessment (FSA) is as high-stakes as they come; students must pass this exam in order to earn their high school diploma. In 2012, the first year of the state-administered Algebra 1 End of Course Exam (EOC), Florida saw high failure rates, with 41% of all middle- and high-school students failing the test. A closer look showed that 52% of ninth-graders failed, and in high-needs schools the failure rate was more than 80%. If it continued, this high failure rate could bottleneck Florida's public school system. This realization highlighted the need for a resource that would help teachers and students alike, thus leading to the development of Algebra Nation.

What is Algebra Nation?

Algebra Nation is a free algebra resource designed to help students succeed in algebra class and on the Algebra 1 FSA exam. Developed by the University of Florida and Study Edge, with funding from the Florida Legislature, this online tool is the first online collaborative community of its kind. The resource ecosystem provides students and teachers with comprehensive support: fun, dynamic standards-aligned concept videos, a web-based practice tool to prepare them for the computer-based test, and easy, seamless access to other students and math experts (called "Study Experts") to receive additional help anytime and anywhere that it is needed.

How Does Algebra Nation work?

Algebra Nation offers free online resources to students and teachers 24/7. Students can log into Algebra Nation through a variety of platforms:

- Web App: www.AlgebraNation.com
- Facebook App: apps.Facebook.com/AlgebraNation
- iPhone and iPad App
- Android App
-

Algebra Nation is now integrated with nearly all of Florida's school districts; therefore students and teachers can log in to Algebra Nation using their existing district credentials. For algebra 1 students, this means they do not have to register or sign up for Algebra Nation – they can get started on AlgebraNation.com at any time. For teachers, this means that student rosters are uploaded for them with assistance from their district technology department, allowing them to access reports on their students' work on Algebra Nation.

Algebra Nation contains both a Student Area and a Teacher Area. The features and benefits of each are described below.

Student Area Features

The **Videos and More** section provides students with instant access at any time of day to a variety of resources. There are two sets of resources: one aligned with the Next Generation Sunshine State Standards (NGSSS), meant for students who took the Algebra 1 EOC before the 2014-2015 school year, and one aligned with the Mathematics Florida Standards (MAFS), for students taking the Algebra 1 FSA during the 2014-2015 school year and beyond. Here, they can watch Study Experts break down concepts through engaging videos, download the companion study guide that allows students to follow along with the videos, and practice additional problems to further improve their understanding of algebra. The practice problems are formatted in a way that strongly parallels what students will see on the Algebra 1 FSA. Funding from the Florida Legislature has also allowed Algebra Nation staff to provide these companion study guides to Florida teachers and students in the form of workbooks. These workbooks, created at teacher request, have been immensely popular. In the last two years, Algebra Nation has sent out nearly half a million workbooks!

On the **Algebra Wall,** Study Experts and teachers award Karma Points to students for helping other students on the Algebra Wall. Points are not awarded for telling another student the answer or for working the problem for that student; instead points are awarded for actions like asking good guiding questions or suggesting a next step. Karma Points are posted on the Leaderboard, and a high number of points can lead to actual prizes like iPads and pizza parties! By harnessing the best aspects of social media and online collaboration, the Algebra Wall provides a safe and fun place for students to help each other and communicate in their native tongue.

The **Test Yourself! Tool** is an interactive platform that provides students with practice problems specifically designed to mirror those on the FSA, including features like drag and drop, equation response, multi-select, graphic response, and open response. Students receive immediate feedback and resources to improve their performance. When a student incorrectly answers a problem, he/she is directed to watch a video on the math concept that was tested, as well as to a solution video showing a classroom teacher breaking down the solution for the missed problem.

Developed by Teachers

Since its inception, the Algebra Nation Student Area has provided over 250,000 students with a platform for social and collaborative learning. Based on the latest research and featuring some of Florida's top mathematics teachers and Study Experts, Algebra Nation provides a dynamic online collaborative learning system for students. From the beginning, Algebra Nation has been conceived, developed, modified, and improved based on feedback from teachers.

Across Florida, teachers have shared with us the many ways in which they have incorporated Algebra Nation in and out of their algebra classrooms. Teachers have indicated that Algebra Nation's universal access and intuitive interface have facilitated the use of such strategies as blended learning, flipped classrooms, and differentiated instruction. Specifically, teachers report using the **Algebra Nation Videos** to introduce a lesson, differentiate instruction by assigning students videos to view based on assessments, reinforce a lesson, as make up work for student or teacher absence, and as professional development (PD) for themselves when they need to review upcoming content. Teachers and administrators report that the MAFS-aligned videos are filling teacher content knowledge gaps as well as introducing them to new methods of teaching the content.

Teachers often assign problems from the **Test Yourself! Practice Tool** as homework, especially since they can check online to see who has completed the assignment. They also use this tool for data collection to document student progress, as formative assessment, as a warm-up activity, as well as for error analysis. As with the videos, teachers also use the Test Yourself! tool as PD as they work to improve their own content knowledge, especially with topics that are new to Algebra 1, like statistics, piecewise functions, and rational exponents. In April 2015, Algebra Nation held our second annual Test Yourself! Challenge, where students from over 50 districts answered over 2 million practice questions in just two weeks. Students who answered at least 80% of the questions correctly were automatically entered into a contest; 100 students won a pizza party for their class and 10 students won an iPad mini. Five teachers received the grand prize of a Bahamas cruise for two!

The **Algebra Nation Wall** provides students with peer tutoring. It is a social collaboration platform where students virtually interact with other students as well as with teachers and Study Experts. On the Algebra Wall, students can ask questions about specific problems either by typing and posting them on the wall, snapping a photo of their homework and uploading on the mobile apps, or using the Draw Something tool to handwrite it. Teachers report that this provides an element of challenge to their high-performing students, who are able to help coach other students from around the state. These high-performing students are learning at the highest levels of Bloom's Taxonomy of Educational Objectives (Anderson et al., 2001), while helping others master the content. Students who need more assistance with algebra are able to learn at their own pace and solicit help in a safe, low-pressure environment. Take a few minutes to read student posts on the wall. As you scroll down, you will perceive the sense of community that has developed among students. It is so gratifying to see a student from one part of Florida offering assistance to a student across the state. Some teachers even use the Algebra Wall as an extension of their office hours for students! The wall also provides teacher PD; reading student posts is a great way to gauge common student misconceptions.

Algebra Nation Teacher Area

Although teachers reported that they were using Algebra Nation as a source of PD in the ways described above, they also wanted help with transitioning to the new MAFS state standards. In 2013 the Bill and Melinda Gates Foundation awarded Algebra Nation a grant to expand our teacher area, to better meet these needs. Since the launch of the Algebra Nation Teacher Area on September 1, 2014, more than 5,500 teachers across Florida have visited the site to access professional development resources that complement the student side of Algebra Nation. This extensive online resource includes several key components, and the entire network can be conveniently accessed 24/7 from any smartphone, tablet, or computer.

Teacher Area Features

To help teachers transition to the new MAFS state standards, the **Videos and More** section provides teachers with carefully curated classroom videos, lesson plans, tasks, and links to additional resources that are aligned with the MAFS Content Standards and Standards for Mathematical Practice. These resources include guided discovery lessons, real-life tasks, graphical interpretations, and rigorous questioning, all with the common focus of improving students' conceptual understanding. There are even NGSSS resources for teachers who are working with students who are re-taking the previous EOC, based on the "old" standards.

The **Teacher Wall** provides mathematics educators with the opportunity to communicate with fellow algebra teachers across the state, share feedback on videos or lessons that they have implemented in their classrooms, and ask for advice on teaching a specific topic. Topics of conversation on the teacher wall range from test item specifications to classroom technology use to differentiated instruction to classroom management advice. Guides (algebra teacher leaders selected

from all over Florida) and Algebra Nation team members are available on the Teacher Wall to support teachers who have questions.

Monthly challenges on the Teacher Wall provide opportunities for teachers to reflect deeply on dilemmas of practice, try new instructional strategies and share the results with others, as well as consider innovative ways to meet the needs of Florida's diverse population of algebra students. Winners of these challenges receive prizes, sparking friendly competition within the Teacher Wall community.

Student Activity Reports provide teachers with information on their students' activities on Algebra Nation, including the number of videos viewed, wall posts, and student scores on the Test Yourself! practice tool. These reports are data points for teachers to use in both formative and summative evaluations of their students' understanding of algebra.

Finally, in response to a clearly stated need among teachers, Algebra Nation has been offering a series of virtual PD events called **Stats Chats**. These weekly collaborative opportunities for 15-25 teachers at a time provide extra support around statistics, an area of mathematics new to the Algebra 1 standards. Facilitated by Algebra Nation Guides, these sessions are recorded and are available for viewing on the website, along with the lessons and/or activities that were shared during the chats. These Stats Chats have become so popular that mathematics departments and mathematics content supervisors across the state use the archived versions as professional development around a topic that is generating great anxiety during this time of transition.

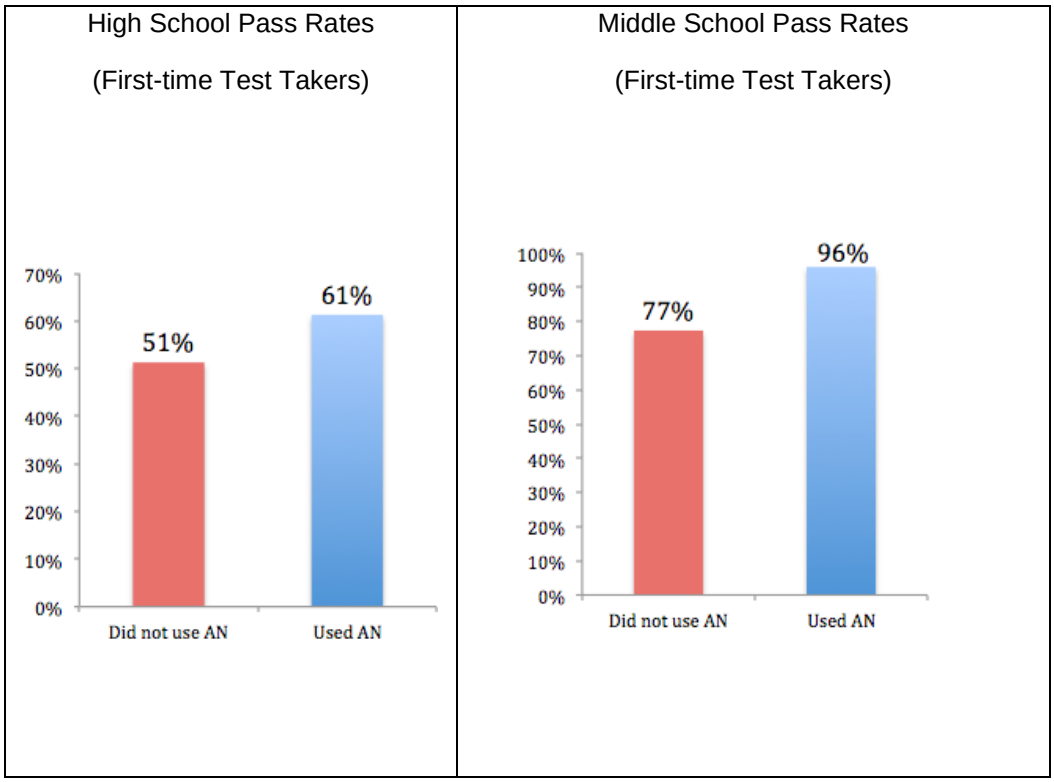
The Power of Algebra Nation

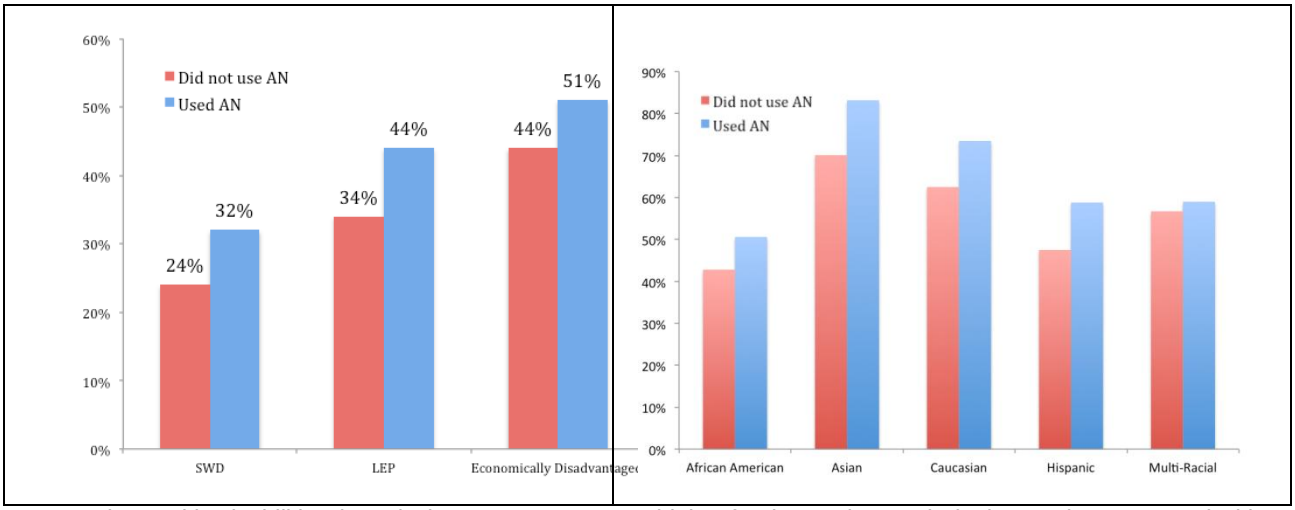
Algebra Nation has a proven track record of improving test scores for ALL students, first-time test takers and retakers alike, regardless of their economic status, level of English proficiency, race or ethnicity. Florida's overall Algebra 1 FSA scores improved 6% since Algebra Nation launched. Algebra Nation districts had a +8.1% greater improvement compared with non-Algebra Nation districts. Algebra Nation districts located in the Florida Panhandle, which are some of the most rural and isolated districts in Florida, performed even better. The Panhandle Algebra Nation districts saw a +13.8% improvement in their Algebra scores relative to the Panhandle districts that did not use Algebra Nation (See Figure 1).

Figure 1: SPRING 2014 – ALGEBRA 1 EOC SCORES ANALYSIS

We looked at a representative sample population of nearly 1,000 students and found that Algebra Nation is effective for students at all ability levels and backgrounds.

High school students who used Algebra Nation had a 10% higher pass rate than those who did not. Middle school students who used Algebra Nation had a 19% higher pass rate than middle school students who did not use Algebra Nation.





For Students with Disabilities (SWD), the pass rate was 8% higher for those who used Algebra Nation compared with those who did not. Students with Limited English Proficiency (LEP) who used Algebra Nation had a 10% higher pass rate than those students who did not use Algebra Nation. For students who are economically disadvantaged, the pass rate was 7% higher for those who used Algebra Nation compared with those who did not. When analyzed by race, students who used Algebra Nation had higher pass rates across the board than those who did not use Algebra Nation, regardless of their race.

Summer 2014 – Algebra 1 EOC Scores Analysis

Students who failed the Spring 2014 Algebra 1 EOC were given the opportunity to retake the test in the Summer 2014 testing session. Districts provide students who failed the Algebra 1 EOC with summer opportunities to prepare for the test. Algebra Nation staff analyzed results from a sample population of students. Students who were exposed to Algebra Nation daily as their primary source of remediation during the entirety of their summer school program had a 50% pass rate on the Summer 2014 Algebra 1 EOC. In comparison, only 33% of the retakers statewide passed the Summer 2014 Algebra 1 EOC.

Algebra Nation Usage

So far, students and teachers have logged in almost 2 million times and posted over half a million posts on the moderated Algebra Wall. Students and teachers have watched nearly 3 million videos, representing more than a million hours of viewership, and over 4 million "Test Yourself!" questions have been answered. There was never a mandate for districts to incorporate or use Algebra Nation, but in less than six months, Algebra Nation was integrated in a vast majority of school districts and available to over 99% of the students who will be taking the Algebra 1 FSA. Currently, over 95% of the schools in Florida who have students taking the Algebra 1 FSA have users accessing Algebra Nation. Algebra Nation has been widely embraced by teachers, students, and administrators alike because of its ease of access, ease of use, and immediate relevance. On Twitter, Facebook, and Instagram, students have called the diverse cast of teachers "cool" on many occasions and considered the videos fun and entertaining.

Virtual Professional Development

Throughout the school year, Algebra Nation provides schools across the state with PD sessions, ranging in length from one hour to full-day trainings. The team members who design and implement this PD have an intimate understanding of Algebra Nation and the Mathematics Florida Standards (MAFS), as well as the FSA test item specifications and limitations. The goal in providing such PD is two-fold: first is to inform mathematics teachers and district personnel about the resources that Algebra Nation offers to both teachers and students, and then to deepen teachers' understanding of the changes in content and pedagogy brought about by the MAFS. Over the course of the 2014-2015 school year, Algebra Nation provided nearly 100 PD sessions, including face-to-face and virtual trainings as requested by teachers, math coaches, and district level math personnel.

Student and Teacher Feedback

In an attempt to improve resources and accessibility to the users, Algebra Nation is constantly seeking feedback from students and teachers alike. When asked about her experience with Algebra Nation in her classroom, an Orange County algebra teacher shared,

> *"When there are so many new and uncertain expectations, it is invaluable to have a reliable and exceptional resource for our students! We and our students feel more prepared and confident as we prepare to take on this year's Algebra 1 EOC."*

Another algebra teacher from Duval County echoed these sentiments:

> *"In my opinion, every algebra teacher should be using Algebra Nation if they want their students to do well on the FSA. Thank you Algebra Nation!"*

Students are also vocal in their appreciation of what Algebra Nation has done to better prepare them for the Algebra 1 EOC. Madison, a student in Orange County, shared her love of

Algebra Nation by writing a poem, which was shared throughout her district. One verse in particular stood out to us as we read her work:

> *"The place I go to ask and be answered, not judged.*
> *To help others understand, to help give them a nudge."*

These words affirm that Algebra Nation is achieving what we initially set out to do: create a safe, collaborative environment where students can go to learn more about algebra and gain the skills, understanding, and confidence they need to succeed on the Algebra 1 FSA.

The Mission

Algebra Nation is continuously seeking to expand and improve its resources for teachers, students, and even parents. The goal is to be a comprehensive 'one-stop-shop' that complements any algebra 1 curriculum being used in the state of Florida. Algebra Nation strives to continuously improve available offerings by eliciting and incorporating feedback from all stakeholders in a timely manner; teachers are often surprised and delighted to see that a suggestion they emailed to the help desk is put into action in a matter of days. Looking to the future of mathematics education in the United States, Algebra Nation understands the importance of providing all students with a rigorous and equitable mathematics curriculum. The free resources that are available at anytime and anywhere for both teachers and students aim to improve the quality of mathematics education by increasing students' understanding of algebra and related concepts, encouraging student and teacher collaboration, and providing content and pedagogical support for mathematics teachers throughout Florida.

Join Algebra Nation

For more information on Algebra Nation, please email help@algebranation.com. To access the Algebra Nation app, search "Algebra Nation" in the App Store or on Google Play.

Reference

Anderson, L.W. (Ed.), Krathwohl, D.R. (Ed.), Airasian, P.W., Cruikshank, K.A., Mayer, R.E., Pintrich, P.R., Raths, J., & Wittrock, M.C. (2001). A taxonomy for learning, teaching, and assessing: A revision of Bloom's Taxonomy of Educational Objectives (Complete edition). New York: Longman.

Dr. Joy Bronston Schackow is a Mathematics Education Professor-in-Residence for the School of Teaching and Learning and the Lastinger Center for Learning at the University of Florida where she works with secondary mathematics teachers from UF partner schools throughout Florida. She has been instrumental in the development of Algebra Nation, and with support from the Bill and Melinda Gates Foundation she co-created the online professional development network area of Algebra Nation to support algebra teachers during the transition to the Mathematics Florida Standards (MAFS).

Stephanie Cugini is a Project Coordinator for Algebra Nation, Her work for the free online program for algebra teachers and students is focused on developing curriculum, facilitating PDs, and increasing awareness of the Algebra Nation throughout the state and country. Stephanie holds a M.Ed. in Curriculum and Instruction from the University of North Florida. Prior to working with Algebra Nation, Stephanie was a high school mathematics teacher in Jacksonville, Florida, where she served as department chair and district curriculum writer.

Kendrick, Michelle W., Elizabeth M. Jakubowski, and Diana Rice
Using Graphing Technology: Strategies for Promoting Student Learning Through Discovery
Dimensions in Mathematics
FALL 2015, 35 (2), p. 13 – 18.

Using Graphing Technology: Strategies for Promoting Student Learning Through Discovery

Michelle W. Kendrick, Elizabeth M. Jakubowski and Diana Rice

The State of Florida requires schools to use a portion of their instructional allocations to purchase digital instructional materials (FS 1006.40, 2014) and there are nationwide initiatives aimed at promoting one-to-one computer allocations for students (Bebell & Kay, 2010). These computing expectations require districts, or schools, to provide all students with access to a computer or tablet at school and home (Penuel, 2006). Districts face a range of issues from identifying the most effective types of technologies to use in their schools (e.g., desktops, laptops, tablets) to determining how best to provide this technology to students, whether through school purchase, lease, or having students provide personal devices. Consider, for example, the case of using graphing calculators in the classroom. Introduced nearly 30 years ago, these devices were often purchased by students individually for one hundred dollars or provided as whole class sets by schools, often at exorbitant costs. Today with tablet or computer access in classrooms for multiple instructional purposes, students might use these devices in place of graphing calculators, which would allow districts to save money on purchases of whole class sets of calculators and reduce pressure on students to purchase their own. Additionally, there is an abundance of free online graphing calculator utilities that students can safely access using the Internet, eliminating the need to buy expensive graphing calculators. Whichever option is elected, calculators or computers, cost is but one issue raised by these mandates.

A more pressing question is how teachers might more effectively incorporate these technologies into their classrooms. Both state and national standards have delineated the importance of using technology to help students develop more conceptual understandings in mathematics. The National Council of Teachers of Mathematics (NCTM) *Standards* (2000) place emphasis on using technology to help students analyze the changes in quantities of linear relationships. In the Florida Mathematics State Standards, there are eight mathematical practices that students should be able to demonstrate when performing mathematics. Several of these practices include constructing viable arguments (justification), modeling, using appropriate tools strategically, and making use of structures (MAFS.K12.MP.3, MAFS.K12.MP.4, MAFS.K12.MP.5, and MAFS.K12.MP.7) (Florida Department of Education, n.d.). One way these mathematical practices may be developed is through student engagement in graphing experiences; wherein students are provided opportunities to explore multiple models and manipulations of equations and inequalities, within a relatively short amount of time.

The purpose of the following discussion is to describe a few ways that teachers may use graphing technology with middle or high school algebra students to address both the NCTM Standards and the Florida Mathematical Practices. The goal is to help teachers engage students in meaningful practice involving the use of graphing technology, whether with graphing calculators or computers. Through this technology, students are able to explore the components of linear equations, inequalities and systems of both. Variations of the ideas presented may also be used to examine absolute value equations, quadratics and many other types of equations.

Linear Equations and Functions

For this first lesson example, students are encouraged to explore the mathematical concept of linear equations using graphing calculators, either hand-held or online (Jarrett, 1998). This activity aligns with MAFS.912.F-IF.3.7, which requires students to "graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases." This standard builds on MAFS.8.F.1.3 requiring students to "interpret the equation $y = mx + b$ as defining a linear function, whose graph is a straight line" (Florida Department of Education, n.d.). Furthermore, this activity aligns with NCTM's Algebra Standards (2000), particularly those specifying that students "explore relationships between symbolic expressions and graphs of lines, paying particular attention to the meaning of intercept and slope" and "use graphs to analyze the nature of change in quantities in linear relationships" (p. 222). While completing this activity, students are able to see how the individual components of a linear equation are structured and affect the direction (positive or negative slope) and steepness of the line. Additionally, through engaging in questioning strategies, students make critical arguments based upon the observation of data (FLDOE, n.d.), which addresses several additional Mathematical Practices.

In this lesson, students may be given a series of linear equations, starting with the most basic equation and then continuing by adding various slopes and/or y-intercepts to alter the given parent function (see Figure 1). For instance, have students begin with $y = x$ and ask them to graph using technology and then sketch that graph on their paper. From this parent graph, have students graph $y = x + 4$, keeping $y = x$ in the function table. Have students sketch the second image and then describe the differences between the two graphs. For instance, students should recognize that the image shifted on the y-axis to intersect at $y = 4$ instead of $y = 0$ and is parallel to $y = x$. Allow students to continue to explore by graphing $y = x - 3$ and describe the changes they see.

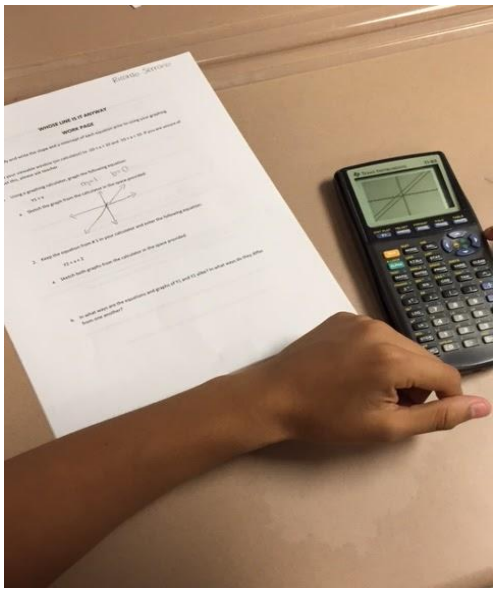


Figure 1

Next, have students graph the parent graph $y = x$ and then $y = 2x$. Ask them to sketch the graph and compare the two images. Ask students to identify what changes occurred when they

graphed $y = 2x$. Students should indicate that they have seen an increase in the steepness of the line. Have students continue this investigation by graphing various equations such as $y = \frac{1}{2}x$, $y = -2x$, and $y = -\frac{1}{2}x$. At each step, require them to sketch their images and describe changes that occur. Finally, have students graph the parent function $y = x$ as well as $y = 2x + 3$ and $y = -2x - 3$ or other combinations of these linear equations, asking them to sketch their images and describe what changes have occurred. During this process students should be discovering that the slope (m or coefficient of x) changes the direction and steepness of the line and that the y-intercept (b or constant) changes where the line intersects the y-axis.

As an assessment for this lesson, teachers should ask students to generalize, without the use of technology, a sketch of a graph given a linear equation or set of equations, such as $y = \frac{1}{2}x - 4$. The students should be able to identify the slope and y-intercept and use this information to accurately sketch a graph of the line.

In this particular lesson, students have learned the concepts of slope and y-intercept, components of the slope-intercept form of a line, and they have learned the basics of graphing linear equations using slope-intercept form in an interactive hands-on lesson that engages them in the learning process. Using intuition, scaffolding, and modeling, students have been given the opportunity to discover the meaning of *slope* and *y-intercept* and how these elements affect the pictorial representation of a linear equation.

Linear Inequalities

With only minor alterations to the previous lesson idea, teachers may use the same concept to allow students to explore linear inequalities and visually recognize how solutions of a linear equation differ from a linear inequality. Whether graphing with calculators or online technology, many graphing tools can now graph inequalities in two variables and the solution(s) of those items. In this lesson, students would not be exploring slope and y-intercept, even though it certainly can be infused as a review, but rather the impact of the inequality symbols (e.g. $<$, $>$, $\leq$, and $\geq$) on the solution.

Using graphing technology, have students graph $y = x$ as the parent graph and sketch the result. Next, have them graph $y > x$ and sketch the image. Ask students to identify and note components that are similar and different. Students should also indicate that shading of a solution area is now present in their graph. Then, ask students to graph $y \geq x$ and ask them to sketch and compare the two graphs. With this comparison be sure that students notice the subtle difference between a *broken* and *solid line* depending on the type of inequality symbol used. The teacher should not provide the answer regarding the difference between the two, but rather ask if anyone notices or has thoughts about why there is a difference. If students do not see the difference, have them experiment with additional graphs such as $y < 2x - 4$ and/or $y \leq 2x - 4$ or similar inequalities. Begin to identify students who recognize the differences and allow these students the opportunity to explain to the class what they have noted about the graphs. Additionally, have students discuss their hypotheses as to why they think some lines appear broken and some solid. In this exploratory effort, it is important that students discover the solutions to these differences in order to develop more conceptually valid understandings.

Quadratic Equations

Again, a version of the original lesson may be used to help students more fully understand quadratic equations. This lesson could certainly serve as an introductory and exploratory lesson that allows students to quickly see the differences between linear equations and quadratic equations. Using the exact steps as in the original lesson, allow students to begin with a parent graph $y = x^2$ and then graph adaptations to this equation. Have students next graph $y = x^2 - 2$ and $y = x^2 + 3$. Have students sketch the three images and describe what changes they see among all three graphs and allow them to begin developing conjectures about simple quadratic equations. Next, have students graph $y = \frac{1}{2}x^2$ and $y = 2x^2$, asking them to sketch the graphs and then to describe the similarities and differences. Finally, give students $y = 3x^2 - 2$ and ask them to hand sketch what they believe the graph should look like on a coordinate plane.

Again, students should be making conjectures throughout the lesson. Certainly, the teacher should add additional equations until students could model appropriately using the graphing technology and explain correctly the components of simple quadratics and how these variations affect the outcome of a graph. Students may then use this knowledge to make conjectures about the sketches of the graphs. Notice that students are using an *inductive* approach to learning many mathematics skills that most would think need to be taught using direct instruction (Jarrett, 1998).

Systems of Equations and Inequalities

The last two lessons described feature using graphing technology to teach systems of equations and inequalities inductively. By now, students are aware of the solutions to linear equations and linear inequalities both through graphing and through solving by hand. As they begin to experiment with systems, some students struggle with the idea of multiple solutions, especially with inequalities. In lessons associated with systems it is more important to show graphically why there may be one solution, no solution, or multiple solutions. It is important that the teacher creates an experience for all three types of situations in order for students to conclude that there will not always be one solution. While it is still important that students learn other methods of solving systems, we feel it is important that they first develop confidence in their ability to identify the number of solutions before encountering situations without context. By constructing meaning in solutions, students can then be confident that any solution they get is potentially appropriate.

In either of these lessons (interchanging inequalities for equations), have students sketch each equation independently. We recommend sketching on patty paper or some other type of tracing paper so that the students may lay one sketch on top of the other and see that they can exist on the same plane together. From here, allow students to use graphing technology to quickly graph several examples $\begin{cases} y = x + 6 \\ y = -0.5x + 3 \end{cases}$, $\begin{cases} 2y - 4x = 2 \\ y = 2x - 4 \end{cases}$ and $\begin{cases} 2y - 4x = 2 \\ y = 2x + 1 \end{cases}$. Have students graph each of these systems using graphing technology and then sketching on paper, comparing the different solutions. The nice thing about using graphing technology is that teachers can use more complex numbers and examples that use fractions or decimals in an effort to vary the student experience. With this approach, students will not be forced to struggle through the arithmetic, as the technology helps in the graphing process, still allowing students to see the effects these numbers have on graphs. Remember, the point of this lesson is not to learn how to

graph, but rather to explore what types of solutions the student can get when working with different systems.

When exploring with systems of inequalities, allow students to experiment with multiple representations, using $<$, $>$, $\leq$, and $\geq$. Let students create their own systems and conjecture solutions based on their previous findings. Students should get the sense that the inequality symbols used have profound effects on the solutions they obtain. Once students have basic knowledge of the types of solutions, they will be better able to solve these systems by hand using multiple methods.

Conclusion

The lessons presented here have been incorporated in middle and high school classrooms and have shown to be effective in helping students learn and retain linear concepts. Specifically, the linear equation lesson was used as part of a small-scale action research project at a developmental research school. The data from this research indicated positive effects upon student learning when using graphing calculators. In the year the lesson was **not** used in teaching linear equations, students scored a class average of 63% on the final unit assessment, with only 16% scoring above 85% on the test and only 47% scoring above 70%. By comparison, the year this lesson was part of students' linear equation learning (primarily as an introductory lesson), they scored a class average of 69% on the final unit assessment with 27% of the students scoring above 85% and 54% above 70%.

All too often students today are receiving direct instruction and being asked to simply memorize rules and algorithms. Whenever there is a possibility for inductive learning, students should be given that opportunity. With the technology now available in most classrooms, teachers can allow their students to experience guided exploration that will engender greater motivation and student ownership in learning. The NCTM Standards (2000) encourage students to use graphs to analyze the changes in quantities in linear relationships and to use modeling to solve problems, all in an effort to create conceptual understanding. What better models are there than those created using graphing technology? Research has shown that in high school algebra classes, graphing technology is a powerful aid in developing understanding (Horak, 1994). Why limit this ability to just high school, when middle school and even elementary teachers can and should be incorporating these experiential learning opportunities into their mathematical studies as well? In evaluating the assessment differences between using and not using graphing technology, one might easily argue that the incorporation of these experiential learning activities would be beneficial for middle and elementary, as well as high school students.

When implementing these lessons, teachers should expect to see an increase in students' engagement, motivation, and understanding and monitor outcomes in all of these areas. Rather than restricting the lessons to equations that the teacher creates, students should be encouraged to experiment with mathematical ideas by developing and exploring their own examples. When students do so and become comfortable enough with a concept to further their own understanding, 'true' learning results.

References

Bebell, D. & Kay, R. (2010). One to one computing: A summary of the quantitative results from the Berkshire Wireless Learning Initiative. *Journal of Technology, Learning, and Assessment, 9*(2), 5-57.

Florida Department of Education (n.d.). *2014 Mathematics Florida Standards*. Retrieved from http://fldoe.org/core/fileparse.php/7575/urlt/mathfs.pdf

Florida Statutes, K-20 Education Code, Support for Learning. Stat. Ann. §§ 1006.07 (2014).

Horak, V. M. (1994). Investigating absolute-value equations with the graphing calculator. *The Mathematics Teacher*; *87*(1), 9-11.

Jarrett, D. (1998). Integrating technology into middle school mathematics: It's just good teaching. *Northwest Regional Educational Laboratory*. Department of Education, Washington, DC.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

Penuel, W.R. (2006). Implementation and effects of one-to-one computing initiatives: A research synthesis. *Journal of Research on Technology in Education, 38*(3), 320 - 348.

Michelle Kendrick (*mkendrick@fsu.edu)* is a doctoral student and research assistant at Florida State University, School of Teacher Education. Her research interests include elementary and middle grades mathematics teaching and learning.

Elizabeth Jakubowski (*emjakubowski@fsu.edu)* is an associate professor of mathematics education at Florida State University, School of Teacher Education. Her teaching and research focus is on the connections between teaching and teacher learning, specifically: knowledge beliefs and practices. She is also focused on making mathematics accessible for students with learning disabilities and teacher education.

Diana Rice (*drice@fsu.edu*) is an associate professor of elementary education at Florida State University, School of Teacher Education. Her research interests include elementary science teaching and learning.

Su, Hui Fang Huang "Angie", Frederick A Ricci, and Mamikon Mnatsakanian
Critical Thinking as a Mathematical Problem Solving Strategy: Classroom Scenario
Dimensions in Mathematics
Fall 2015, 35 (2), .p. 19 – 26.

Critical Thinking as a Mathematical Problem-Solving Strategy: Classroom Scenario

Hui Fang Huang "Angie" Su, Frederick A. Ricci, and Mamikon Mnatsakanian

TWO WRONGS SIMPLY MAKE A RIGHT PYTHAGOREAN TRIPLE

> Of fate two wrongs infer one right (C. Ackers for J. Wilford, ed. 1734. "The London Magazine, or, Gentleman's Monthly Intelligencer, Volume 3")

Critical thinking in Mathematical problem-solving

Educators from kindergarten through college often stress the importance of teaching critical thinking within all academic content areas (Foundation for Critical Thinking, 2007, 2013). As indicated by the position statements of the National Council of Teachers of Mathematics, high quality mathematics education before the first grade should use curriculum and teaching practices that strengthen children's problem-solving and reasoning processes as well as representing, communicating, and connecting mathematical ideas" The joint position statement of NAEYC and the National Council of" (NAEYC & NCTM [2002] 2010, 3).

Through the educational and academic institutions critical thinking is identified as an important outcome for achieving the higher orders of learning upon successful completion of a course, a promotion, or a degree (Humphreys, 2013; Jenkins & Cutchens, 2011). Although there are numerous definitions of critical thinking, the authors have selected the definition by Scriven & Paul, 2008 as "the intellectually disciplined process of actively and skillfully conceptualizing, applying, synthesizing, and/or evaluating information gathered from, or generated by, observation, experience, reflection reasoning, or communication as a guide to belief and action" (Scriven & Paul, 2008). Instructors should teach problem solving within the context of mathematics instruction and engage students in critical thinking by thoughtful questions with discussion of alternative results. Teaching preschool children to problem solve and engage in critical thinking in the context of mathematics instruction requires a series of thoughtful and informed decisions.

MAKING THE CASE IN THE CLASSROOM: An actual historical classroom example

The student, called Mike, made two mistakes, forgetting to extract a square root, adapting to innovation, and flexibility as students usually do. Nevertheless, he immediately realized how to fix them. The result was a simple way to construct Pythagorean triples with an insightful geometric mnemonic rule. We present the following story for accuracy of its **mathematical** content.

Ms. Angel asked her students to find at least one more Pythagorean triple besides 3, 4, and 5. Nobody came up with one, so Ms. Angel asked Mike to come to the board. She knew Mike was a skilled student, and with her guidance he would be able to construct one such triple.

"Let's try to find a Pythagorean triple, a right triangle with integer sides," Ms. Angel said.

Mike was completely confused. He could not remember mathematical concepts well, but utilizing his critical thinking skills, he was not too embarrassed to ask questions: "Why did you call them Pythagorean?" Mike asked.

"They obey the Pythagorean Theorem," Ms. Angel responded.

Mike continued asking, "What was the Pythagorean Theorem about?"

"The square of the hypotenuse equals the sum of the squares of the legs."

Mike was not satisfied with the explanation, so he remarked, "This is complicated."

"Well, we have the simplest example of a Pythagorean triple: 3, 4, 5," said Ms. Angel.

"That's really nice. What about 1, 2, 3? May I draw it?"

"You may try, but it's not going to be a Pythagorean triple."

"Why not?"

"Because there is no such right triangle with sides 1, 2, 3. It's not even a triangle, but three overlapping segments."

"What about 4, 5, 6?"

"No again, it's not a right triangle."

"How should I know that?"

"Because 4 squared plus 5 squared does not equal 6 squared."

"I know how to square numbers, I just draw a square, with that number of unit squares on the sides, and count the number of unit squares inside it." Mike eagerly showed what he knew.

"That's very good, Mike!" Ms. Angel commented. "Back to Pythagorean triples. Do you know another example besides the 3, 4, and 5?"

"Um, let me try. I draw two legs, say, 1 and 2. I square them, 1 and 4, and then add, making 5. That's the hypotenuse of the Pythagorean 3, 4, 5 triangle."

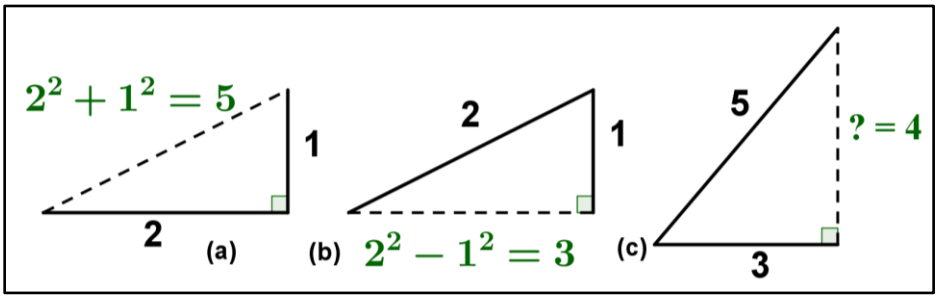


Figure 1

"Not really. You forgot to take the square root; the hypotenuse is a square root of 5 which is not an integer. So, try other possible legs."

Embarrassed, Mike replied, "Sorry, I forgot about the square root."

By now, Mike got really curious and asked, "What if I take the difference of the squares that we already have? The difference of 4 and 1 is 3. That's the leg!" Mike exclaimed.

"Again, you forgot to take the square root!"

"But look, earlier I found the hypotenuse, 5. And now I found the leg 3 of the same 3, 4, 5 Pythagorean triple."

"What about the other leg?" Ms. Angel pressed on.

"I don't know...isn't it automatically determined?"

"Yes it is. But it may not give you a Pythagorean triple ... hmm ... but wait a minute. Obviously, it is 4," Ms. Angel replied.

"Are you sure? That's really cool!"

"That's very interesting! You started with two wrong legs, 1 and 2, and obtained two new correct legs, 3 and 4," smiled Ms. Angel.

"But I had 5 before, so I created the 3, 4, 5 Pythagorean triple by starting with 1 and 2."

"Yes, but this was just a coincidence. Do you want to try another example?"

Mike got really interested and said, "Okay, give me two numbers for the legs."

"Take 3 and 4."

"But, we just did it."

"No, you started with 1 and 2. Now I want you to start with 3 and 4 and go on your way. You will definitely end up with the hypotenuse not being 5, because 5 is the hypotenuse of the 3, 4, 5 triangle. Your hypotenuse is 25, which is 3 squared plus 4 squared."

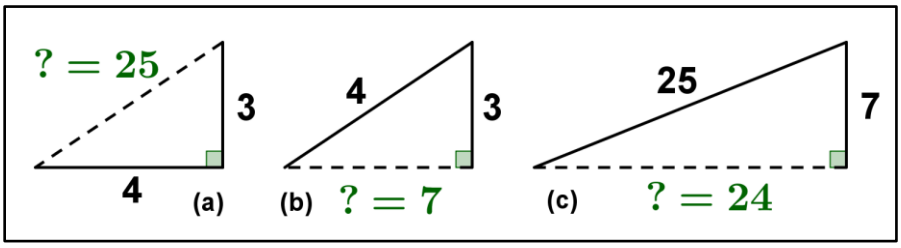


Figure 2

"And that will be wrong?"

"You got it!"

"I am sure you are right, but let me understand…. 25 was the sum of 3 and 4 squared."

"That's correct, Mike!"

"Well, 3 squared is 9, and 4 squared is 16; the sum is 25. But, now I will take their difference which is 16 - 9 = 7. The leg is 7, and not 3 or 4."

"What do you mean, Mike? Aren't you constructing the 3, 4, 5 triangle?"

"No, I am not."

"Then what is the second leg, Mike?"

"We should apply the Pythagorean Theorem."

"Let's try it: $25^2 - 7^2 = 625 - 49 = 576$," Ms. Angel helped to calculate.

"That's too large for a leg," announced Mike.

"We didn't finish yet, Mike. Take the square root ... WOW!"

"Is there something wrong?"

"You are fortunate; it's 24, a correct value. But something is very wrong here."

Beginning to feel comfortable, Mike suggested to Ms. Angel, "Let's try another example, say, 2 and 5. Square them: 2 squared is 4, and 5 squared is 25."

"The sum is 29, that's the hypotenuse," Mike continued. "The difference is 21; that's a leg. The other leg is determined. In fact, I know how to find it. I just double the product of my initial legs, twice 2 times 5 gives 2×2×5=20," Mike wrote on the board. "That's the second leg!"

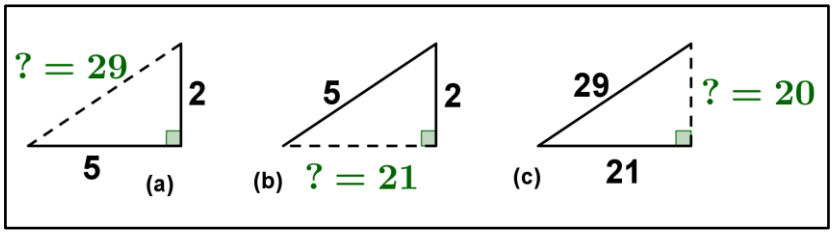


Figure 3

"How did you come up with the double legs product, Mike?"

"Well, I need the difference of two squares. I factored it in my head into their sum and difference. I knew that one of them is a sum and the other is a difference; I could visualize the terms canceling each other and obtaining the product of doubled squares. Because each term is a square of the starting numbers, I saw the product as the double product squared. I don't know how to extract square roots, but I know that the square root of a squared whole number is exactly that number itself."

"That sounds reasonable. Let me check: $20^2 + 21^2$ is 841; that is, 29^2. That's correct, Mike."

"Ms. Angel, I think I did a good job with my homework, right?"

"No, you didn't. If you did, you would not have made this cute discovery."

"What discovery? Isn't this known?"

"The result is known, but not your approach! You made a mistake, or guess, and then fixed it. And you did that not just once, but twice!"

"How many Pythagorean triples are there, Ms. Angel?"

"That number is doubly infinite."

"Can I get all of them?"

"Using your method? Yes, you can!"

"That sounds great!"

"The question is how to prove that your method always works, Mike!"

"We just check the result." Mike replied after a short pause.

"Checking is not a proof in mathematics, although…"

"I mean, checking the general case, for any two starting whole numbers for legs."

"You just gave an idea for a proof. Take two arbitrary integers, call them m and n. The hypotenuse is m squared plus n squared. Take m > n. Then one of the legs is m-squared minus n-squared. Now, if you square these two values and subtract, you will get the

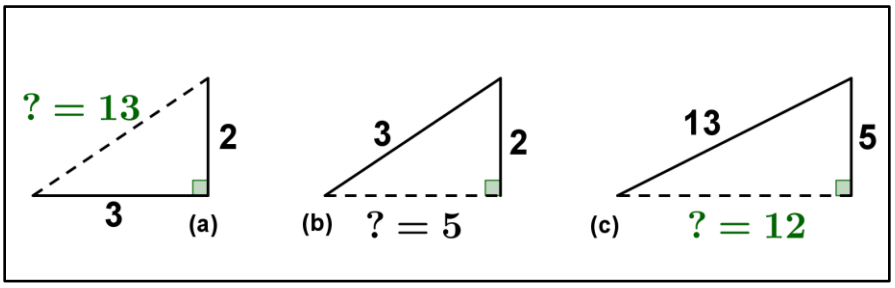


Figure 4

square of the doubled product, 2mn. This proves the sufficiency of your method." Ms. Angel continued, "I don't know how to prove the necessity, but, I know that this is a known Euclid's two-parametric representation for Pythagorean triples."

"I was sure this is known."

"But your method is also very educational and fun! Isn't it?" She asked the class.

"Yes, it is!" The entire class shouted. "What we like the most is the clearer visual geometric arguments," several students commented. "It helps to remember what to do."

"Then, try this at home. Last time no one came up with any new Pythagorean triple; but now I am asking you to create a dozen of your own Pythagorean triples. It will take less than an hour; it's such an easy homework assignment!" replied Ms. Angel. "I will check the dozen of the Pythagorean triples known to the ancient Babylonians. Some of them involve huge numbers. I was always curious how they obtained these. With guessing and checking it seems impossible, but with Mike's guessing and fixing it's very possible."

"We can construct really large Pythagorean triples," commented some students.

Ms. Angel continued, "I am very pleased with today's work. I also learned some other things that I never knew before: *In Mike-Euclid's Pythagorean triples, the hypotenuse itself is a sum of two square integers*; besides the fact that its square is a sum of squares of two integers."

Then she added. "I didn't realize before that also in *Mike-Euclid's Pythagorean triples, one of the legs itself is a difference of two square integers*; besides the fact that its square is a difference of squares of two integers."

"The nice thing is that no square root need be extracted." Mike commented proudly.

Ms. Angel decided to summarize the discovery; "We still have ten minutes before the end of class. Let me sketch on the board the summary of the things we learned today. And please make your notes in the notebooks."

Summary: Pairs of wrongs make all right Pythagorean triples.

1. Start with any two integers, and square them.

2. Add the squares to get the hypotenuse, but forget to take the square root (Fig. 5a).

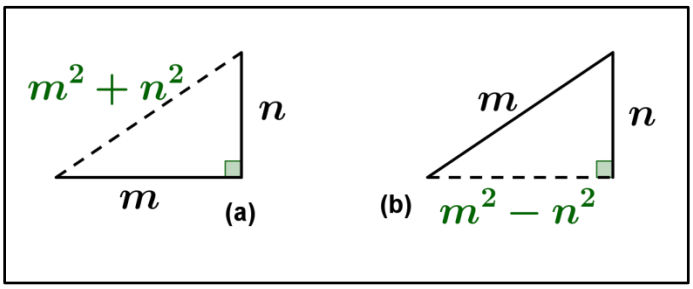


Figure 5

3. Subtract the two squares to get a leg, but forget to take the square root (Figure 5b).

4. The other leg is determined automatically. It is the double product of the starting numbers.

5. For a proof, check the algebra with general notations, using factoring (Fig. 6).

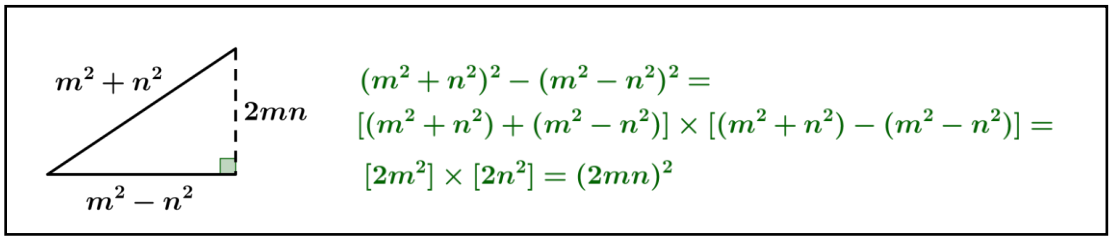


Figure 6

6. A pseudo-geometric interpretation makes it easy for us to remember the Euclid's algorithm.

7. The three sides in Fig. 6, together with their multiples, generate all the Pythagorean triples.

Concluding remarks:

Sixty years later Mike extended his insightful geometric vision with doing wrongs and taking no roots to higher dimensional spaces, thus generating correct Pythagorean n-tuples, quadruples, pentuples, sextuples, etc. Their known standard formulas are too complicated to memorize. However, the critical thinking technique used by the teacher provides the heuristic teaching approach, which trains the student's mind to become an independent thinker.

This scenario included examples of all the critical thinking factors discussed and needed for embracing new thinking for problem-solving, which is a basis for self-empowerment and enhancement of leadership in all nations. Mathematics teachers can change lives by assisting students to become critical thinkers/problem solvers who will be ready to assume the roles of future leaders, change, and innovation in our expanding global networked society. At an increasing speed, globalization is changing work settings and nonwork environments, and it demands new leaders to make decisions and solve problems often and quickly. Chartrand, Ishikawa, and Flander (2009) cited The Conference Board studies, which indicated that 70% of employees with a high school education were deficient in critical thinking skills and even 10% of college graduates lacked critical thinking skills.

There are many activities that demonstrate opportunities to utilize questioning and critical thinking skills within the mathematical courses of instruction. This real-case scenario was written to provide an example on how teachers can incorporate critical thinking into lesson plans, curriculum and classroom activities.

References

Chartrand, J., Ishikawa, H., & Flander, S. (2009). Critical thinking means business. *Learn to apply and develop new, 1*. (Retrieved from http://138.9.110.12/Documents/ business/techservices/ThinkWatson-Critical%20Thinking-Means-Business.pdf

Foundation for Critical Thinking. (2007). To analyze critical thinking, we must analyze and question its elemental structures. Reference materials from *The Thinker's Guide to Analytic Thinking*. Retrieved from http://www.criticalthinking.org/ctmodel/logic-model1.htm

Foundation for Critical Thinking. (2013). Critical thinking: Where to begin. Retrieved from http://www.criticalthinking.org/pages/critical-thinking-where-to-begin/796

Humphreys, D. (2013). Employers more interested in critical thinking and problem solving than college major. Retrieved from aacu.org

Jenkins, D. M., & Cutchens, A. B. (2011). Leading critically. *Journal of Leadership Education, 10*(2), 1-21. Retrieved from https://www.academ75ia.edu/1214625/

Scriven, M., & Paul, R. (2008). Defining critical thinking: *Foundation for Critical Thinking*. Retrieved from http://www.criticalthinking.org/aboutCT/ourConceptCT.cfm

---

Hui Fang Huang "Angie" Su (shuifang@nova.edu). Angie is a Professor of Mathematics Education for the Fischler College of Education's Graduate Teacher Education Program at Nova Southeastern University. She is the creator of Project MIND - Math Is Not Difficult®, a K-12 mathematics enhancement project currently being implemented in hundreds of classrooms throughout the US. She has received numerous awards and recognitions, including the *Presidential Award for Excellence in Mathematics and Science Teaching* from the National Science Foundation. Su was the recipient of several research awards. Her research interests include mental calculation, algebraic thinking, elementary number theory, integrating technology into the teaching and learning of mathematics, distance learning and education, gifted and special education, and creating games that will make learning difficult mathematical concepts easy to understand.

Fredrick Ricci (fredricc@nova.edu). He is currently Professor of Organizational Leadership and Human Resource Development at Nova Southeastern University. He has held several international positions in the area of business and education. Professor Ricci has served in the U.S. Air Force and was an Instructor with the U.S. Navy PACE program on military bases and U.S. naval ships.

Mamikon A. Mnatsakanian is Project Associate at Project Mathematics! at California Institute of Technology. He received a Ph.D. in physics in 1969 from Yerevan State University, where he became professor of astrophysics. As an undergraduate he specialized in the development of geometric methods for solving calculus problems by a visual approach that makes no use of formulas, which he later developed into his system of **visual calculus**.

In 2010 he was nominated by Caltech for the Ambartsumians International Prize, awarded annually by the President of Armenia, for his contributions in the field of theoretical astrophysics.

In 1959 he discovered a new proof of the Pythagorean Theorem.

Harshman, Katie and Heidi Eisenreich
Instructional Strategies that Foster Conceptual Understanding
Dimensions in Mathematics
FALL 2015, 35 (2), p. 27 – 32.

Instructional Strategies that Foster Conceptual Understanding

Katie Harshman and Heidi Eisenreich

Mathematics educators agree that a conceptually based approach to instruction is beneficial to students (NRC, 2001). Although there has been a call for a reform in the teaching of mathematics for decades (NCTM, 1989, 1991, 2000), there has not been a great deal of change in teaching practices in the United States (NRC, 2001). If research shows that conceptual instruction is more beneficial than direct instruction, why do we as teachers continue to lecture to our students? It can be argued that teachers simply teach the way they were taught. In addition, pre-service teachers enter universities with certain beliefs and attitudes not in line with conceptual instruction techniques that facilitate making sense of problems (Stohlmann, Cramer, Moore, & Maiorca, 2014). This could be a result of pre-service teachers, in their own schooling experience, being taught rules and procedures. It is often difficult to convince in-service teachers to instruct mathematics *conceptually* rather than *procedurally*.

It could be argued that with instruction focused on procedures without meaning, teachers assume that students have mastered prior knowledge necessary to solve the mathematics, which is not always the case. This is a disadvantage for students who might have solved a problem differently if they had been given the chance. It also forces students to be passive learners, dependent on the teacher, and the likelihood that students will attempt to solve problems without step-by-step instructions is minimal. When students are given the opportunity to construct their own understanding of a mathematical concept, they are more likely to comprehend the material conceptually and achieve mathematical proficiency (Cobb, 1988; NRC, 2001).

We as teachers want our students to succeed and are responsible for helping them reach mathematical proficiency. Although teaching for conceptual understanding is more beneficial for students, some teachers do not know how to properly implement it into their classrooms. A common unfortunate scenario is that students are put into groups with the expectation that they will work together, however the teacher ends up lecturing while students sit together and take notes. This article aims to aid teachers in the implementation of instruction that fosters their students conceptual understanding of mathematics.

Suggested Best Teaching Practices for Conceptual Understanding

- It is effective to have students working in collaborative groups rather than alone or in pairs, as this encourages student discourse and discovery rather than the teacher demonstrating what they need to do. Be forewarned however, that cooperative grouping is not simply having students push their desks together. Students need to be taught how to work in this manner, and teachers should be cognizant of selecting **appropriate tasks** that foster collaborative work (NRC, 2001).
- It is best for teachers to present students with a word problem, rather than a number sentence. The benefits of teaching through contextual problems are that students are open to solve the problem any way they like, and real world situations better foster a student's thinking and knowledge (Streefland, 1991).

- As students are working together, the teacher can visit groups to help facilitate the discussion. Having students explain and justify their reasoning promotes conceptual understanding. Also, getting away from the front of the room and interacting with students while they work is a chance to assess student understanding (NRC, 2001).
- Once all groups are finished and have agreed upon an answer, the teacher can ask for volunteers to share. This is a good time for the teacher to facilitate meaningful whole group discussions. Encouraging students to participate in mathematical discourse is part of the NCTM Principles to Actions: Ensuring Mathematical Success for All (NCTM, 2014).
- In order to help students achieve the correct solution, teachers should be aware of typical student errors (Ashlock, 2010). Even if there are no errors, the teacher could write an incorrect answer on the board and tell the class she saw a group give that answer. This could lead to a discussion on why the answer is incorrect, in which case students would be engaging in CCSS Standard for Mathematical Practice 3: Construct Viable Arguments and Critique the Reasoning of Others (CCSSI, 2010).

A Snapshot of a Lesson on Fraction Subtraction (Third Grade)

The teacher presents the following problem for students on the board: The cake batter machine broke after it had made $2\frac{1}{2}$ pounds of batter. The baker used $\frac{4}{5}$ of a pound. How many pounds of batter was not used?

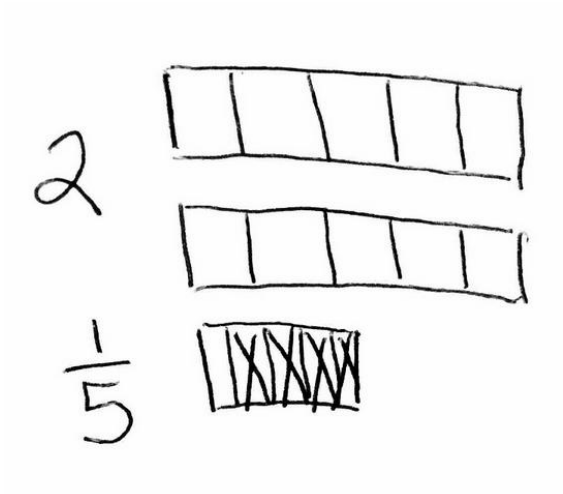
Teacher: (Reads problem for students) "Ok everyone please work in your groups to find the solution to the problem."

Teacher circles the room to make sure everyone is off to a good start. She stops at one group and everyone has $2\frac{1}{5}$ on their paper.

Teacher: "I see you already have an answer. Can someone explain how you got it?"

Jessica: "Yes, so we drew $2\frac{1}{2}$, since that's what we started with, then since we were taking away $\frac{4}{5}$, we knew we had to change our pieces into fifths. So we cut everything into 5 pieces. Then we took away 4 of them. We had 2 wholes left and $\frac{1}{5}$ of a whole left."

Jessica's drawing:



Teacher: “Alex, you don’t look happy with this answer. Do you agree?”

Alex: “Well, it kind of makes sense when Jessica explained it, but I thought all the pieces had to be the same size before you could subtract. Like, the pieces in the bottom row are smaller.”

Teacher: “Hmm, Jamie what did Alex just say?”

Jamie: “He said the pieces in the bottom row are different sizes, and now I’m not sure my answer is right.”

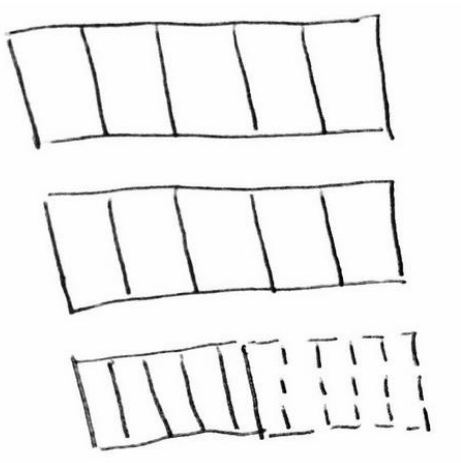
Teacher: “Jessica what do you think?”

Jessica: “They are different sizes. I don’t think we did it right.”

Teacher: “What do your pieces on the bottom row represent?”

Alex: (Draws the remaining whole for the third row and cuts it into pieces) “If I draw the rest of the whole here, since I had 5 pieces in my half, I need 5 more to make a whole. So I have ten pieces. Those are tenths. So we have tenths in the bottom row and fifths in the top two rows.”

Alex’s drawing:



Teacher: “Ok, so is there a way to make them all the same size?”

Jessica: “Yes! We could cut the fifths in half!”

Teacher: “How is that helpful?”

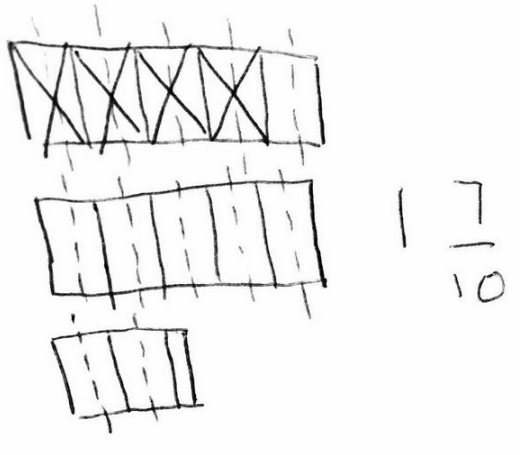
Jessica: “Because when you cut pieces in half, the total number out of the whole is doubled, so instead of fifths we will have tenths.”

Teacher: “I think you’re on to something. Keep going and see what answer you get.”

Teacher moves on to another group. They all have $1\frac{7}{10}$ on their papers.

Teacher: “I see you guys are already finished. Can you explain how you got your answer?”

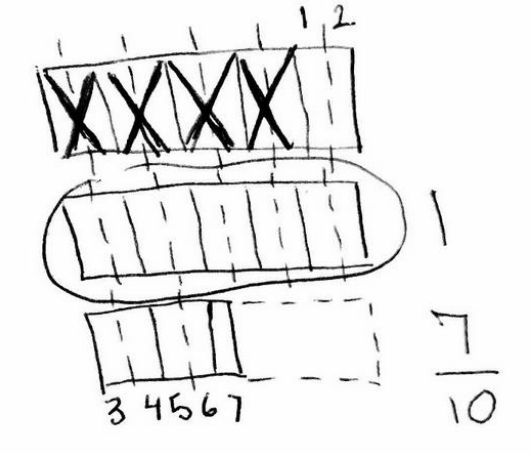
Jacob's drawing:



Jacob: "I'll explain. So, we started by drawing the $2\frac{1}{2}$ to represent what the baker had to start with. Then we knew we had to subtract $\frac{4}{5}$ since that's how much of $2\frac{1}{2}$ he used. So we tried cutting everything into fifths but we realized we would have half of a piece in our last row. That was a problem. So, since that piece was already cut in half we decided to cut everything else in half, and we ended up with tenths. Then we took away $\frac{4}{5}$ and ended up with $1\frac{7}{10}$."

Teacher: "Tava can you explain how you guys took away fifths and ended up with tenths?"

Tava's drawing:



Tava: "Well, we didn't really take away $\frac{4}{5}$ we took away $\frac{8}{10}$."

Teacher: "How do you know?"

Tava: "Because when we cut our fifths in half, we ended up with tenths. So $\frac{1}{5}$ is the same as $\frac{2}{10}$, and we took away 4 of them, so we took away $\frac{8}{10}$."

Teacher: "So you're saying that $\frac{4}{5}$ is different than $\frac{8}{10}$?"

Tava: "Yes."

Eddie: "No they're the same."

Teacher: "How do you know?"

Eddie: "Because if you just look at the drawing we took away the same amount."

Teacher: "How do you know it's the same amount? Use your own words."

Eddie: "Well, I guess we started out with fifths, but when we cut them in half we had tenths. So we started with taking 4 fifths away, but we cut the pieces in half. The pieces got smaller and were doubled, so we were really taking away 8 pieces out of the ten that made up the whole."

Teacher: "Ashley do you agree?"

Ashley: "Yes they're the same because if you have 4 pieces out of 5, and you cut everything in half, everything gets doubled. So $\frac{8}{10}$ is the same is $\frac{4}{5}$."

Teacher: "Do we all agree?"

Everyone nods.

Teacher leaves and goes around to check on other groups. She is satisfied with their answers and heads to the front of the room.

Teacher: "Wow you guys really did a great job working together to solve this problem. I saw a lot of different strategies out there. Does someone want to come up and show the class how your group solved it?"

At this point the teacher can have several groups share their answers and their peers can ask them questions. This is beneficial because students can see that even though multiple representations of the same problem were presented and discussed, everyone arrived at the same correct solution. This would also be a good time for the teacher to bring up other solutions or common errors. For example the teacher might say, "I saw one group with an answer of $\frac{17}{10}$ pounds. Is this correct?" or "I saw another group with an answer of $2\frac{1}{5}$ pounds. Is this correct?" Then the teacher can lead fruitful discussions regarding why $\frac{17}{10}$ is the same as $1\frac{7}{10}$, and why $2\frac{1}{5}$ is incorrect. These discussions are important as they foster the conceptual understanding of the mathematics that new standards require.

In the above classroom, the teacher acted as a facilitator rather than lecturer. Posing contextual problems allows students to learn mathematics through real world situations. This allows students to build meaning for the mathematical concepts, which aids in the progression to more abstract concepts (Van de Walle, Karp, & Bay-Williams, 2013). Once students can make a connection between concepts and algorithms, mathematical proficiency has been achieved. (Carpenter, 1986; NRC, 2001). A teacher's instruction is the catalyst for a student's conceptual comprehension of mathematics. Teachers should not distress if they only get through a few problems with their students. Students will have a deeper understanding of the mathematics if they work more deeply and thoroughly on a few problems. This is more advantageous then completing, say, 30 of the same type of problem using rules and procedures without understanding why those procedures work.

This type of instruction is not easy to implement, but with practice and perseverance, students will get accustomed to the routine and actually enjoy having mathematical debates with their classmates. This type of classroom is beneficial because the teacher holds all students accountable for participating and explaining their classmates' answers. It is also beneficial for students who may have a difficult time sitting through a lecture because students are more involved with their learning by talking, drawing, and debating with their classmates. It is through these important discussions that teachers can be sure their students have achieved mathematical proficiency.

References

Ashlock, R. B. (2010). *Error patterns in computation: using error patterns to help each student learn* (10th ed.). Boston, MA: Pearson.

Carpenter, T. P. (1986). Conceptual knowledge as a foundation for procedural knowledge. In J. Hiebert (ed.), *Conceptual and Procedural Knowledge: The Case of Mathematics*. Hillsdale, NJ: Erlbaum.

Cobb, P. (1988). The tension between theories of learning and instruction in mathematics education. *Educational Psychologist, 23*(2), 87.

Common Core State Standards Initiative (CCSSI). (2010). Common Core State Standards for Mathematics. Washing- ton, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf

National Council of Teachers of Mathematics (NCTM). (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: Author

National Council of Teachers of Mathematics (NCTM). (1991). *Professional standards for teaching mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics (NCTM). (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics (NCTM). (2014). *Principles to actions: Ensuring mathematical success for all*. Reston, VA: NCTM

National Research Council. (2001). *Adding it up: Helping children learn mathematics.* J. Kilpatrick, J. Swafford, and B. Findell (Eds.). Mathematics Learning Study Committee, Center for Education, Division of Behavioral Sciences and Education. Washington, DC: National Academy Press.

Stohlmann, M., Cramer, K., Moore, T., & Maiorca, C. (2014). Changing pre-service elementary teachers' beliefs about mathematical knowledge. *Mathematics Teacher Education & Development, 156.* Retrieved from http://www.merga.net.au/ojs/index.php/mted/article/view/208

Streefland, L. (1991). *Fractions in realistic mathematics education: A paradigm of developmental research*. Dordrecht: Kluwer.

Van de Walle, J. A., Karp, K. S., & Bay-Williams, J. M. (2013). *Elementary and middle school mathematics: teaching developmentally* (8th ed.). Boston: Pearson.

Katie Harshman (Katie.Harshman@knights.ucf.edu) is a doctoral student in mathematics education at the University of Central Florida.

Heidi Eisenreich (Heidi.Eisenreich@knights.ucf.edu) is a doctoral candidate in mathematics education at the University of Central Florida.

Mark your 2016 calendar!

Save the Dates!

Florida Council of Teachers of Mathematics
64th Annual Conference

Imagine the Possibilities…

to Infinity and Beyond!

October 20 – 22, 2016
Doubletree by Hilton Orlando at SeaWorld

For details see the FCTM website (FCTM.net)

Mathematics Educator of the Year
Kenneth P. Kidd Award

Mathematics educators are eligible if they meet the following criteria, as established by the FCTM Board of Directors.

1. The person shall have been actively involved in mathematics education in Florida for at least ten (10) years.
2. The person has been professionally active and has made a significant contribution at the local, state, and/or national level.

For the nominee, please send the following information:

a. Name, home address, and phone number
b. Work address and phone number
c. Educational Background
d. Experience in the field of mathematics education
e. Professional activities
f. Your name and address.

Nomination forms are available on the web site: www.fctm.net

For additional information contact: Susan Lee, FCTM Grants and Awards Chair
Rebecca Lee @ ocps.net

Completed Packets are due by April 15, 2016.

Mathematics Educators of the Year
Past Kenneth P. Kidd Award Recipients

1976 Mary Nesbit, Dade County, FL
1977 Ruth Bower, West Palm Beach, FL
1978 George Rule, Orlando, FL
1979 Baker Holman, Pensacola, FL
1980 Mary Elizabeth Sullivan, Dade County, FL
1981 Ernest Burgess, Boca Raton, FL
1982 JoAnne Taber, Miami, FL
1983 Arthur T. Minor, Palm Harbor, FL
1984 Bill E. Jordan, Winter Park, FL
1985 Richard A. deAguero, Miami, FL
1986 Herbert Johnson, Clearwater, FL
1987 Laurence R. Wantuck, Hollywood, FL
1988 Dr. Betty Lichtenberg, Temple Terrace, FL
1989 Dr. Elroy Bolduc, Gainesville, FL
1990 Dr. Douglas Brumbaugh, Chuluota, FL
1991 Dr. Tom Denmark, Tallahassee, FL
1992 Dr. Donovan Lichtenberg, Temple Terrace, FL
1993 Dr. Charles Andy Reeves, St. Petersburg, FL
1994 Renee Henry, Tallahassee, FL
1995 Dr. Don Bernard, Gainesville, FL
1996 Margaret Hackworth, Pinellas County, FL
1997 Dr. Charles Aplin, Ft. Walton, FL
1998 Nicky Walker, Milton, FL
1999 Barbara Nunn, Coral Springs, FL
2000 Suzanne Davis, Largo, FL
2001 Carol Newman, Plantation, FL
2002 Roger L. O'Brien, Bartow, FL
2003 Karen R. Swick, Lake Worth, FL
2004 Dr. Denisse Thompson, Lutz, FL
2005 Dr. Karol Yeatts, Miramar, FL
2006 Susan McNally, Ft. Myers, FL
2007 Carol Martin, Lake City, FL
2008 Carole Halka, Ft. Lauderdale, FL
2009 Debbie Gillis, Okeechobee County, FL
2010 Ann Marie Hubscher Rockledge, FL
2011 Nancy Kinard, Tequesta, FL
2012 Dr. Lou Cleveland, Marianna, FL
2013 Charlene Kincaid, Milton, FL
2014 Jill Nielsen, New Port Richie, FL
2015 Joyce Hawkins, Hallandale Beach, FL

Applications for all **Grants and Awards** are now accepted online
Please visit the **Grants & Awards** section of our web site: www.fctm.net

Tom Denmark Teacher Enhancement Grant

In the 1990's the FCTM Board established four annual grants in recognition of Dr. Tom Denmark, a long time active FCTM member. Through these grants, the Board hopes to encourage participation at annual FTCM conferences as a way to provide excellent professional development for FCTM members. Enhancement Grants for up to four teachers to attend the annual FCTM Conference will be awarded, one each to a Primary (Pre-K – 2) teacher, an Intermediate (3-5) teacher, a middle School (6-8) teacher and a high school (9-12) teacher. **The annual deadline for applications is April 15**. Recipients of the awards will be notified in late May of their selection. They will be required to submit a written report on their session and workshop experiences encountered at the Annual Conference along with their receipts for reimbursement, no later than November 30. The FCTM treasurer will send a check to each recipient upon receipt of these materials.

Each recipient of this grant may be reimbursed up to $500. This will help to cover the cost of registration, room and transportation to attend the Annual Conference. Any additional monetary expenses over the $500 grant that are incurred by the recipient will be at their own expense. It is hoped that if added funding is necessary the recipient may be successful in seeking other support.

** This grant money may not be used for FCTM membership dues, food or substitute pay.

Who May Submit an online Application:

1. Any full time classroom teacher currently certified and teaching in Florida may apply.
2. Applicants must have at least a bachelor's degree.
3. Applicants must be teaching mathematics at the time of the application and anticipate teaching mathematics during the following year.
4. Applicants must be current members of FCTM.
5. Current FCTM Board members and past recipients may not apply.

Kenneth Kidd Grant

FCTM sponsors mini-grants of up to $500 for projects to improve mathematics education

The Board is looking for high quality projects that will improve mathematics education at any level. Projects which have been funded in the past include writing guidelines for competitions at the elementary level, developing monthly activities for each grade level in an elementary school, organizing community resources, etc. Project ideas will be judged on originality, uniqueness, and the potential impact on teachers and students.

Any FCTM member with **four years of experience in the State of Florida** may apply. The annual deadline for applications is **April 15**. The person receiving the grant will submit a written reflection about the project's implementation and success, to be considered for publication in FCTM's journal or newsletter.

Apply online with the following information:

1. Experience teaching mathematics (beginning with the current year), School, level, Dates
2. Narrative (500 words maximum) to include
    a. specific objective(s)
    b. description of products (if any)
    c. how you will determine the success of your grant
3. Budget Outline including proposed timeline for expenditure of grant funds
4. Name of reference supporting the grant award (principal, department chair. Mathematics supervisor, college professor). Your reference may be contacted for verification.

Lichtenberg Pre-service Educator Grant

Any Florida college or university full-time, pre-service educator in his /her junior or senior year may apply for this grant. **The annual deadline for application is September 1**. The grant will reimburse awardees up to $500 for transportation, room and registration expenses to attend the FCTM Annual Conference in the fall. Pre-service educators who receive this grant will be required to write about their experiences in a form suitable for one of FCTM's publications prior to reimbursement. An electronic copy of the manuscript and receipts for reimbursement must be submitted to the FCTM treasurer no later than December 1. Up to four grants may be awarded each year. It is expected that all awardees will be from different institutions.

Apply online and respond to the following:

1. Describe the type of teacher of mathematics (Pre-K to 12) you would like to become.
2. Explain how attending the FCTM Conference will help you attain that goal.
3. How will you share what you have learned with other pre-service mathematics educators at your college/ university?

Don Bernard Enhancement Grant

In 2010 the FCTM Board established two annual grants in recognition of Dr. Don Bernard, a long time active FCTM member who supported mathematics teachers in a variety of ways. Through these grants, the Board hopes to encourage participation at annual FCTM conferences as a way to promote mathematics professional development for FCTM members who provide support to mathematics classrooms though not a full time classroom teacher.

Two grants will be awarded to attend the annual FCTM Conference. The annual deadline for applications is April 15. Recipients of the awards will be notified in late May of their selection. They will be required to submit a written report on their session and workshop experiences encountered at the Annual Conference along with their receipts for reimbursement, no later than November 30. The FCTM treasurer will send a check to each recipient upon receipt of these materials.
Each recipient of this grant may be reimbursed up to $500. This will help to cover the cost of registration, room and transportation to attend the Annual Conference. Any additional monetary expenses over the $500 grant that are incurred by the recipient will be at their own expense. It is hoped that if added funding is necessary the recipient may be successful in seeking other support.
** This grant money may not be used for FCTM membership dues, food or substitute pay.

Who May Submit an online Application:

1. Anyone who provides support to mathematics classrooms though not a full time classroom teacher.
2. Applicants must have at least a bachelor's degree and hold a valid Florida teaching certificate.
3. Applicants must be current members of FCTM.
4. Current FCTM Board members and past recipients may not apply.

Florida Council of Teachers of Mathematics
Membership Application
(Individual or Affiliate Group)

Florida Council of Teachers of Mathematics (FCTM) – Check membership option and amount.

_____ **One** Year Membership - $25.00

_____ **Two** Year Membership - $45.00

_____ **Five** Year Membership - $100.00

_____ Full-time Student **One** Year Membership - $12.50
(also send an official class schedule)

Write your check for the appropriate amount, payable to **FCTM**, and mail the check and this form to:

Diane Gard
FCTM Membership
PO Box 411884
Melbourne, FL 32941-1884

Please complete the following; help us keep our records up to date.

NAME _____

MAILING ADDRESS _____

CITY_____**STATE**_____ **ZIP** _____

PREFERRED TELEPHONE NUMBER _____

PREFERRED EMAIL _____

SCHOOL / ORGANIZATION_____

COUNTY FOR VOTING PURPOSES _____

Please be sure to provide an e-mail address so that your membership can be confirmed and your Region Director can contact you.

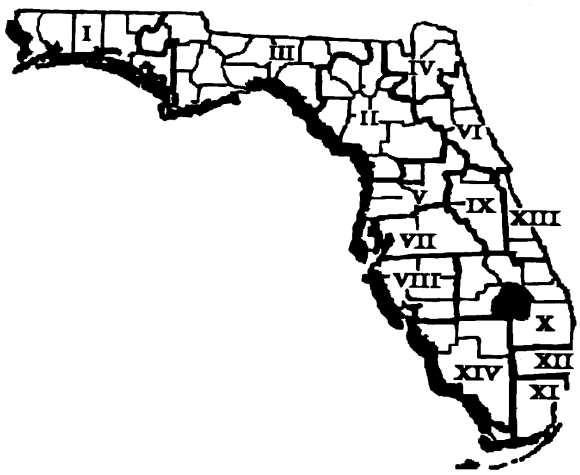
NOTE: Photos taken by authorized FCTM agents of activities and participants during FCTM conferences and meetings are the property of FCTM. Photos may be used for promoting or displaying FCTM activities in print or digital media.

2015 Region Directors

I. ***Amelia McCurdy***
mccurdya@mail.santarosa.k12.fl.us
Bay, Escambia, Holmes, Okaloosa, Santa Rosa, Walton

II. ***Paige Allison***
pallison@pky.ufl.edu
Alachua, Citrus, Columbia, Dixie, Gilchrist, Lafayette, Levy, Marion, Suwannee, Union

III. ***Rob Schoen***
rschoen@lsi.fsu.edu
Calhoun, Franklin, Gadsden, Gulf, Hamilton, Jackson, Jefferson, Leon, Liberty, Madison, Taylor, Wakulla, Washington



IV. ***Claire Riddell***
riddellc@duvalschools.org
Baker, Bradford, Clay, Duval, Nassau

V. ***Lisa Robertson***
robertsonl@pcsb.org
Hernando, Lake, Pasco, Pinellas, Sumter

VI. ***Margaret Bambrick***
mbambric@volusia.k12.fl.us
Flagler, Putnam, St. Johns, Volusia

VII. ***Joseph Ratasky***
joseph.ratasky@sdhc.k12.fl.us
Hillsborough, Polk

VIII. ***Jayne Arrington***
bartandjayne@gmail.com
Charlotte, DeSoto, Hardee, Manatee, Sarasota

IX. ***Lisa Greco***
grecoli@osceola.k12.fl.us
Orange, Osceola, Seminole

X. ***Loryn Lenartowicz***
loryn.lenartowicz@palmbeachschools.org
Glades, Highlands, Martin, Okeechobee, Palm Beach, St. Lucie

XI ***Chris Ruda***
cruda@juno.com
Miami-Dade

XII ***Joan O'Brien***
joan.obrien@ browardschools.com
Broward

XIII ***Leslie Kraynik***
kraynik.leslie@brevardschools.org
Brevard, Indian River

XIV ***Jennifer Edwards***
jenniferae@leeschools.net
Collier, Lee, Hendry, Monroe

✂ cut here

www.fctm.net

Joe Smith
City, Florida

Membership Card
Valid Through 1/1/2012

✂ cut here

Dimensions in Mathematics
c/o Rick Austin,
17413 Heather Oaks Pl
Tampa, FL 33647